第二讲
相对论力学和电动力学

Relativistic Mechanics and Electrodynamics

6.3 相对论理论的四维形式

1、洛伦兹变换的四维形式

洛伦兹变换形式为

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y, \quad z' = z$$

$$\xrightarrow[(x,y,z,ict)]{\text{闵可夫基四维坐标}}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

令$x_1 = x, x_2 = y, x_3 = z, x_4 = ict,$

则上式可写为

$$\boxed{x_\mu' = a_{\mu\gamma} x_\gamma} \longrightarrow \text{洛伦兹变换}$$

其中$x_\mu' = a_{\mu\gamma} x_\gamma \equiv \sum_\gamma a_{\mu\gamma} x_\gamma = a_{\mu 1} x_1 + a_{\mu 2} x_2 + a_{\mu 3} x_3 + a_{\mu 4} x_4$

令：$A=\begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{bmatrix}$

矩阵A为洛伦兹变换矩阵

A的转置是A的逆矩阵 $\tilde{A}=A^{-1}$或$\tilde{A}A=I, I$为单位矩阵

$$a_{\mu\lambda}a_{\mu\gamma}=\delta_{\lambda\gamma} \qquad \delta_{\lambda\gamma}=\begin{cases}1, \lambda=\gamma \\ 0, \lambda\neq\gamma\end{cases}$$

A是正交矩阵,洛伦兹变换是正交变换

洛伦兹变换的反变换

$$\left.\begin{aligned} x'_\mu &= a_{\mu\gamma} x_\gamma \\ a_{\mu\lambda} a_{\mu\gamma} &= \delta_{\lambda\gamma} \end{aligned}\right\} \qquad a_{\mu\lambda} x_\mu{}' = x_\lambda$$

例1.利用洛伦兹变换证明间隔不变性。

证明:

$$S^2 = x^2 + y^2 + z^2 - c^2 t^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_\lambda x_\lambda$$

同理: $S'^2 = x'_\mu x'_\mu$

有 $x'_\mu x'_\mu = a_{\mu\gamma} x_\gamma a_{\mu\lambda} x_\lambda = \delta_{\lambda\gamma} x_\gamma x_\lambda = x_\lambda x_\lambda$

即 $S'^2 = S^2$,可见间隔在洛伦兹变换下是不变的。

2、物理量按洛伦兹变换性质分类

（1）标量　物理量在洛伦兹变换下不变的量

例：间隔$S^2 = x_\mu x_\mu$，$\mathrm{d}S^2 = dx_\mu dx_\mu$为标量

（2）矢量　与坐标有同样变换关系的量

$$g'_\mu = a_{\mu\gamma} g_\gamma$$

（3）张量　张量的变换关系为 $T'_{\mu\gamma} = a_{\mu\lambda} a_{\gamma\rho} T_{\lambda\rho}$

举例计算：偏导运算变换 $\dfrac{\partial}{\partial x'_\mu}$

达朗贝尔算符□= $\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial}{\partial t^2}$

固有时的微分形式 $d\tau = \dfrac{1}{c} dS$

四维速度矢量U_μ $$U_\mu = \frac{dx_\mu}{d\tau}$$

$$d\tau = dt\sqrt{1-\frac{u^2}{c^2}} \Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u$$

$$\boxed{U_\mu = \gamma_\mu(u_1, u_2, u_3, ic)}$$ 四维速度分量

$$U_\mu = \frac{dx_\mu}{dt}\cdot\frac{dt}{d\tau} = u_\mu\frac{1}{\sqrt{1-u^2/c^2}} = \gamma_u u_\mu$$

其中$u_\mu = \dfrac{dx_\mu}{dt}$, ($\mu$=1, 2, 3) 为通常意义下的速度

参考系变换时， $U'_\mu = a_{\mu\gamma}U_\gamma$

3、多普勒效应与光行差公式

$\Sigma \to \omega,\ \mathrm{k}$

$\Sigma' \to \omega',\ \mathrm{k}'$

（1）四维波矢量 k_μ 四维相位 ϕ

四维相位 $\phi = \vec{k}\cdot\vec{x} - \omega t$ 是标量

$$\Rightarrow \boxed{k_\mu = (k_1, k_2, k_3, \frac{i\omega}{c}) = (\vec{k}, \frac{i\omega}{c})}$$

满足洛伦兹变换 $k'_\mu = a_{\mu\nu} k_\nu$

（2）多普勒效应 推导

$$\boxed{\omega' = \omega\gamma(1 - \frac{v}{c}\cos\theta)}$$

若Σ'为光源的静止参照系，则 $\omega' = \omega_0, \omega_0$ 为静止光源的辐射角频率。

$$\boxed{\omega = \frac{\omega'}{\gamma(1-\frac{v}{c}\cos\theta)} = \frac{\omega_0\sqrt{1-v^2/c^2}}{(1-\frac{v}{c}\cos\theta)}}$$

相对论的多普勒效应公式

当 $v<<c$ 则

$$\omega \approx \frac{\omega_0}{1-\frac{v}{c}\cos\theta}$$ 经典多普勒效应公式

若迎着光源运动方向观察辐射，由于$\theta=0$则

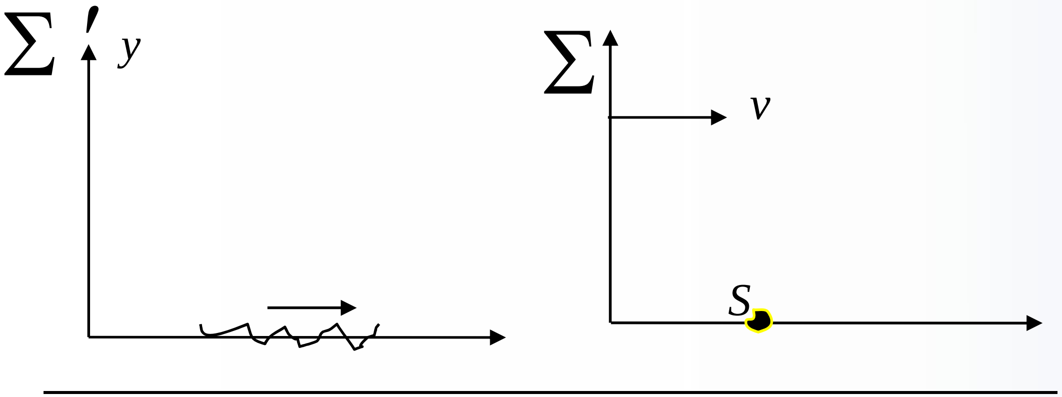
$$\omega = \omega_0\sqrt{\frac{1+v/c}{1-v/c}}$$ 显 然 $\omega > \omega_0\ (v << c)$

这种现象称为纵向多普勒效应

若在垂直于光源运动方向观察，$\theta=\frac{\pi}{2}$ 则

$$\omega=\omega_0\sqrt{1-(v/c)^2}$$ 显然$\omega<\omega_0\left(v<<c\right)$

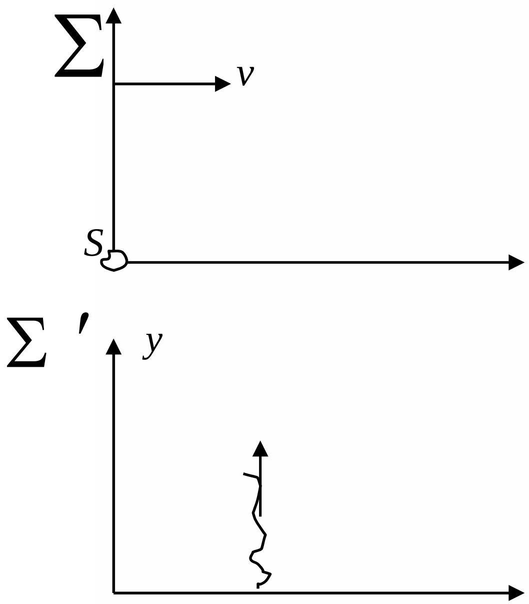
这种现象称为横向多普勒效应



纵向多普勒效应

当 $\theta = 0^o$

$$\omega = \omega_0 \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} > \omega_0$$



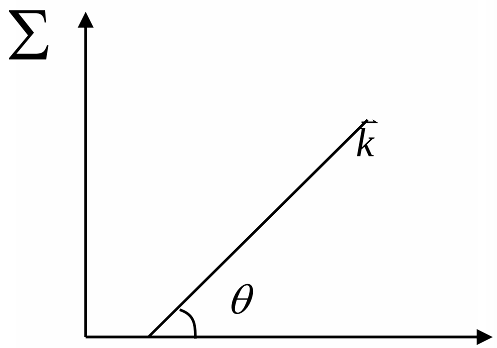
横向多普勒效应

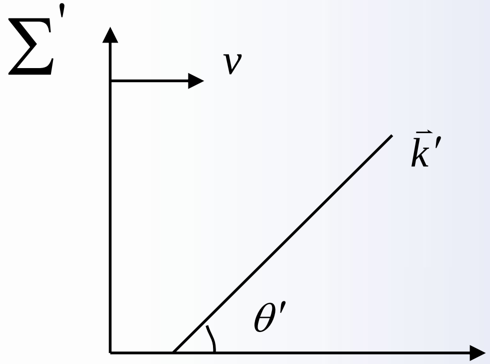
当 $\theta = 90^o$

$$\omega = \omega_0 \sqrt{1 - (v/c)^2} < \omega_0$$

（3）光行差公式

光行差代表在不同的参考系观察光的传播方向之间的关系。





$$\left.\begin{aligned} k_1' &= a_{11}k_1 + a_{14}k_4 = \gamma\left(k_1 - \frac{v\omega}{c^2}\right) \\ k_1 &= k\cos\theta = \frac{\omega}{c}\cos\theta \\ k_1' &= \frac{\omega'}{c}\cos\theta' \end{aligned}\right\} \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \frac{v\omega}{c^2}\right)$$

即： $\omega^{'}\cos\theta^{'}=\gamma\left(\cos\theta-\frac{v}{c}\right)\omega$

$k_2'=k_2 \qquad (\mu=2)$

$k_3'=k_3 \qquad (\mu=3)$

$$\boxed{\omega'=\omega\gamma(1-\frac{v}{c}\cos\theta) \quad (\mu=4)}$$

$$\left\{\begin{aligned}\cos\theta' &= \frac{\cos\theta-\frac{v}{c}}{1-\frac{v}{c}\cos\theta} \\ \sin\theta' &= \frac{\sin\theta\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta}\end{aligned}\right.$$

光行差公式

$$\boxed{tg\theta'=\frac{\sin\theta}{\gamma(\cos\theta-\frac{v}{c})}}$$

还可以通过速度变换公式求得

4、物理规律的协变性

相对论原理要求一切惯性参考系都是等价的。在不同惯性系，物理规律应该表现为相同的形式，即物理规律的协变性。

例如在

Σ 系中 $\quad F_\mu = G_\mu$

$\Sigma \rightarrow \Sigma' \quad F_\mu' = a_{\mu\nu} F_\nu = a_{\mu\nu} G_\nu = G_\mu'$

Σ'系 $\quad F_\mu' = G_\mu'$

协变性

6.3 电动力学理论的协变性

1、四维电流矢量与连续性方程的协变性

（1）四维电流密度矢量

$$U_\mu = \gamma_u(\vec{u}, ic)$$

$$\gamma_u = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\longrightarrow \quad j_\mu = \rho_0 U_\mu$$

ρ_0为标量，静止电荷密度

$$\Rightarrow \quad \begin{cases} \vec{j} = \dfrac{\rho_0 \vec{u}}{\sqrt{1-v^2/c^2}} = \rho \vec{u} \\ j_4 = ic\dfrac{\rho_0}{\sqrt{1-v^2/c^2}} = ic\rho \end{cases} \quad \Rightarrow \quad \boxed{j_\mu = (\vec{j}, ic\rho)}$$

$\longrightarrow$ $\rho = \dfrac{\rho_0}{\sqrt{1-\beta^2}}$ 运动电荷电量

这说明电流密度和电荷密度是个统一物理量。

在Σ'系里观察，电荷静止，只有电荷密度ρ_0；在 Σ系观察，电荷运动，既有电荷密度ρ又有电流密度$\vec{j}$。

（2）连续性方程的协变性

$$\boxed{\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0} \xrightarrow[x_\mu = (\vec{x}, ict)]{j'_\mu = a_{\mu\gamma} j_\gamma} \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$

2、四维势矢量和达朗贝尔方程的协变性

达朗贝尔方程

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}$$

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}$$

$$\xrightarrow{\Box\equiv\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}$$

$$\Box\vec{A}=-\mu_0\vec{j}$$

$$\Box\varphi=-\frac{\rho}{\varepsilon_0}$$

洛伦兹条件 $\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0$

$$\boxed{\frac{\partial A_\mu}{\partial x_\mu}=0}$$

引入A_4

$$\Box\varphi=-\frac{\rho}{\varepsilon_0}\xrightarrow{\mu_0\varepsilon_0=\frac{1}{c^2}}=-\mu_0c^2\rho\xrightarrow{j_\mu=(\vec{j},ic\rho)}\Box\frac{i\varphi}{c}=-\mu_0 j_4$$

定义 $A_4=\frac{i\varphi}{c}$ 则 $\boxed{\Box A_\mu=-\mu_0 j_\mu}$

$$\boxed{A_\mu=(\vec{A},\frac{i}{c}\varphi)}$$

四维矢量

$$x_\mu = (\vec{x}, ict) \qquad k_\mu = (\vec{k}, \frac{i\omega}{c})$$

$$j_\mu = (\vec{j}, ic\rho) \qquad U_\mu = \gamma_\mu(\vec{u}, ic) \qquad A_\mu = (\vec{A}, \frac{i}{c}\varphi)$$

矢量变换为 $\boxed{g'_\mu = a_{\mu\nu} g_\nu}$

$$(a_{\mu\upsilon}) = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

3、电磁张量和麦克斯韦方程的协变性

在电磁场中矢量 $\vec{E}$ 和磁感应强度 $\vec{B}$ 可以由矢势 $\vec{A}$ 和标势 φ 得到

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}$$

其分量为

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \qquad E_1 = ic\left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}\right)$$

$$B_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \qquad E_2 = ic\left(\frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}\right)$$

$$B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \qquad E_3 = ic\left(\frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4}\right)$$

引入一个四维张量

$$F_{\mu\nu}=\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}$$

$$F_{\mu\nu}=-F_{\nu\mu}$$

$$F_{\mu\nu}=\begin{bmatrix} 0 & B_3 & -B_2 & \frac{-i}{c}E_1 \\ -B_3 & 0 & B_1 & \frac{-i}{c}E_2 \\ B_2 & -B_1 & 0 & \frac{-i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix}$$

电场 $\vec{E}$ 和磁场 $\vec{B}$ 是 张量 $F_{\mu\nu}$ 的不同分量

$$\left.\begin{aligned} &\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \mu=4 \\ &\nabla\times\vec{B}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \mu=1,2,3 \end{aligned}\right\}$$

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 j_\mu}$$

推导

$$\left.\begin{aligned}&\nabla\cdot\vec{B}=0 \qquad (1,2,3)\\&\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad (2,3,4)(3,4,1)(4,1,2)\end{aligned}\right\}$$

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}=0}$$

推导

对于 $\dfrac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 j_\mu$

当 $\mu=4$

$$\frac{\partial F_{41}}{\partial x_1}+\frac{\partial F_{42}}{\partial x_2}+\frac{\partial F_{43}}{\partial x_3}=\mu_0 j_4$$

$$\frac{\partial \frac{i}{c}E_1}{\partial x_1}+\frac{\partial \frac{i}{c}E_2}{\partial x_2}+\frac{\partial \frac{i}{c}E_3}{\partial x_3}=\mu_0 ic\rho$$

$$\nabla\cdot\vec{E}=\mu_0c^2\rho=\frac{\rho}{\varepsilon_0}$$

当 $\mu=1$

$$\frac{\partial F_{12}}{\partial x_2}+\frac{\partial F_{13}}{\partial x_3}+\frac{\partial F_{14}}{\partial x_4}=\mu_0 j_1$$

$$\frac{\partial B_3}{\partial x_2}-\frac{\partial B_2}{\partial x_3}-\frac{\frac{i}{c}\partial E_1}{ic\partial t}=\mu_0 j_1$$

$$(\nabla\times\vec{B})_1-\frac{1}{c^2}\frac{\partial E_1}{\partial t}=\mu_0 j_1$$

对于 $$\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}=0$$

当μ,ν,λ为1,2,3

$$\frac{\partial F_{12}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}=0$$

$$\frac{\partial B_3}{\partial x_3}+\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}=0$$

即： $\nabla\cdot\vec{B}=0$

当μ,ν,λ为2,3,4

$$\frac{\partial F_{23}}{\partial x_4}+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}=0$$

$$\frac{\partial B_1}{ic\partial t}+\frac{-\frac{i}{c}\partial E_3}{\partial x_2}+\frac{\frac{i}{c}\partial E_2}{\partial x_3}=0$$

$$\frac{\partial B_1}{\partial t}+\frac{\partial E_3}{\partial x_2}-\frac{\partial E_2}{\partial x_3}=0$$

即： $(\nabla\times\vec{E})_1=-\frac{\partial B_1}{\partial t}$

张量变换：

$$\boxed{F'_{\mu\nu}=a_{\mu\lambda}a_{\nu\rho}F_{\lambda\rho}}$$

电磁场变换规律

电磁场变换规律

E

$$F_{41}' = a_{41}a_{14}F_{14} + a_{44}a_{11}F_{41`} \quad \rightarrow E_x' = E_x$$

$$F_{42}' = a_{41}a_{22}F_{12} + a_{44}a_{22}F_{42`} = \frac{F_{42} - i\beta F_{12}}{\sqrt{1-\beta^2}} \quad \rightarrow E_y'$$

$$F_{43}' = a_{41}a_{33}F_{12} + a_{44}a_{33}F_{43`} = \frac{F_{43} - i\beta F_{13}}{\sqrt{1-\beta^2}} \quad \rightarrow E_z'$$

B

$$F_{23}' = a_{22}a_{33}F_{23} \quad B_1' = B_1 \rightarrow B_x$$

$$F_{12}' = a_{11}a_{22}F_{12} + a_{14}a_{22}F_{42`} = \frac{F_{12} + i\beta F_{42}}{\sqrt{1-\beta^2}}$$

$$F_{31}' = a_{33}a_{11}F_{31} + a_{33}a_{14}F_{34`} = \frac{F_{31} + i\beta F_{34}}{\sqrt{1-\beta^2}}$$

$$E_1' = E_1$$

$$E_2' = \gamma(E_2 - vB_3)$$

$$E_3' = \gamma(E_3 + vB_2)$$

$$B_1' = B_1$$

$$B_2' = \gamma(B_2 + \frac{v}{c^2}E_3)$$

$$B_3' = \gamma(B_3 - \frac{v}{c^2}E_2)$$

$\vec{E},\vec{B}$ 是统一的物理量，在一个惯性系中，若只有静止的电场 $\vec{E}$，则另一个相对运动的惯性系中，则 $\vec{E},\vec{B}$ 同时存在。

上式可以写成更紧凑的形式

$$E'_{//} = E_{//}$$

$$E'_{\perp} = \gamma(\vec{E} + \vec{v}\times\vec{B})_{\perp}$$

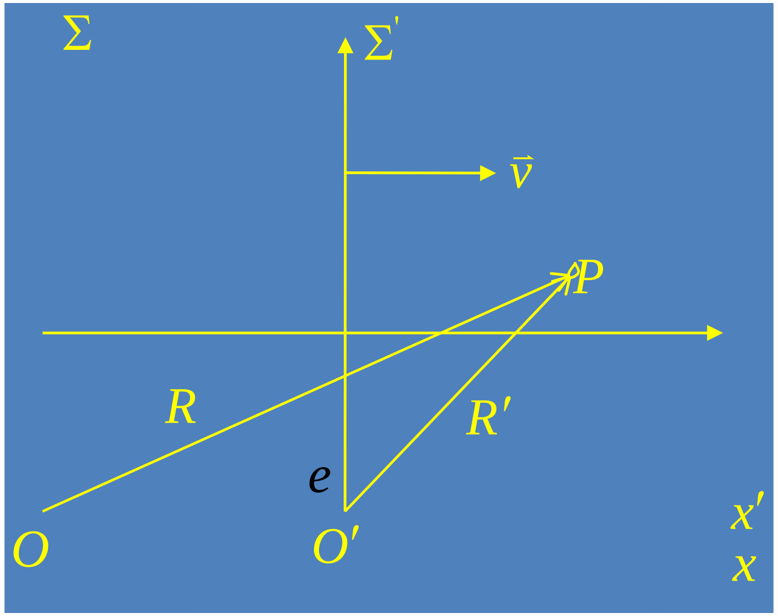
$$B'_{//} = B_{//}$$

$$B'_{\perp} = \gamma(\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E})_{\perp}$$

$v << c$

$$\Longrightarrow \quad \vec{E}' = \vec{E} + \vec{v}\times\vec{B},\quad \vec{B}' = \vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}$$

例1



解：在Σ'系中，粒子静止

$$\varphi' = \frac{1}{4\pi\varepsilon_0}\frac{e}{R'} \qquad \vec{A}' = 0$$

利用反变换关系

$$A_\mu = a_{\gamma\mu} A'_\gamma$$

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \frac{i\varphi}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{i}{c}\varphi' \end{pmatrix}$$

$$A_1 = a_{11}A_1' + a_{41}A_4' = -i\beta\gamma\frac{i\varphi'}{c} = \frac{\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{4\pi\varepsilon_0}\frac{e}{R'} \quad (1)$$

$$A_2 = 0, \quad A_3 = 0$$

$$A_4 = \gamma A_4' \Rightarrow \qquad \varphi = \frac{1}{\sqrt{1-\beta^2}}\varphi' = \frac{1}{4\pi\varepsilon_0}\frac{\frac{e}{R'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$R' = \sqrt{x'^2 + y'^2 + z'^2}$是$\Sigma'$系的距离，可以用$\Sigma$系中的距离$R$代替。

由于洛伦兹收缩 $x = x'\sqrt{1-\beta^2} \qquad y' = y, \qquad z' = z$

$$\Rightarrow \quad R' = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{\frac{x}{\sqrt{1-\frac{v^2}{c^2}}} + y^2 + z^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\sqrt{R^2(1-\frac{v^2}{c^2}) + \frac{(\vec{x}\cdot\vec{v})^2}{c^2}}$$

代入（1）得：
$$A_1 = \frac{\mu_0}{4\pi}\frac{ev}{\sqrt{R^2(1-\frac{v^2}{c^2}) + \frac{(\vec{x}\cdot\vec{v})^2}{c^2}}}$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{e}{\sqrt{R^2(1-\frac{v^2}{c^2}) + \frac{(\vec{x}\cdot\vec{v})^2}{c^2}}}$$

其中 $\vec{x}$ 是 P 点在 Σ 系的坐标矢量，

$R = |\vec{x}|$ 是原点到 P 点的距离，

$$R = \sqrt{x^2 + y^2 + z^2}$$

6.6 相对论动力学

1、四维动量矢量

固有时间隔

固有长度间隔

四维速度矢量 $U_\mu = \gamma_u(\vec{u}, ic)$ $\gamma_u = \dfrac{1}{\sqrt{1-u^2/c^2}}$

四维动量矢量 $\vec{P} = m_0 U_\mu$ $\boxed{P_\mu = \left(\vec{P}, \dfrac{iW}{c}\right)}$ m_0是静止质量

空间分量

$$\vec{P} = m_0\gamma_u\vec{u} = \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} \xrightarrow{u << c} \vec{P} = m_0\vec{u}$$

经典动量

$$P_4 = m_0 i\gamma_\mu c = \frac{i}{c}\frac{m_0c^2}{\sqrt{1-u^2/c^2}} = \frac{iW}{c}$$

这里$W = \dfrac{m_0c^2}{\sqrt{1-u^2/c^2}}$，是相对论中的能量

$$(1-\frac{u^2}{c^2})^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{u^2}{c^2}+\cdots$$

$$\Rightarrow W = m_0c^2+\frac{1}{2}m_0u^2+\cdots$$

定义静止能量

$$W_0 = m_0c^2$$

由此可见，相对论能量**W**不仅包括物体的动能**T**还包括静止能量**W_0=mc^2**，物体静止时仍然有能量。能量守恒定律——利用原子能的理论根据。

物体的动能为 $T = W - W_0 = \dfrac{m_0c^2}{\sqrt{1 - u^2/c^2}} - m_0c^2$

$$\boxed{\text{四维动量}\quad P_\mu = \left(\vec{P}, \frac{iW}{c}\right)}$$

P_μ 可构成不变量, $P_\mu P_\mu = \vec{P}^2 - \dfrac{W^2}{c^2} =$ 不变量

在物体静止内，$\vec{P} = 0$， $W = W_0 = m_0c^2$， 因而不变量为

$-\dfrac{\left(m_0c^2\right)}{c^2} = -m_0^2c^2$， 代入上式得$\vec{P}^2 - \dfrac{W^2}{c^2} = -m_0^2c^2$， 则有

$W = \sqrt{P^2c^2 + m_0^2c^4}$

这是关于物体的能量、动量和质量的一重要公式。

2、质能关系

由相对论能量、动量表示为

$$W=\frac{m_0c^2}{\sqrt{1-u^2/c^2}}\text{(质能关系式)},\ \vec{P}=\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}$$

定义运动质量$m=\dfrac{m_0}{\sqrt{1-u^2/c^2}}$，则有$W=mc^2$，$\vec{P}=m\vec{u}$

3、相对论力学方程

四维力矢量 $$K_\mu=\frac{dP_\mu}{d\tau}=\left(\vec{K},K_4\right)$$

四维力矢量的空间分量 $$\vec{K}=\frac{d\vec{P}}{d\tau}=\frac{d\vec{P}}{\sqrt{1-\beta^2}dt}$$

$$\vec{K}\sqrt{1-\beta^2}=\frac{d\vec{P}}{dt},\quad \vec{P}=\frac{m_0\vec{u}}{\sqrt{1-\beta^2}}$$

定义力：$\vec{F}=\sqrt{1-\beta^2}\,\vec{K}=\sqrt{1-u^2/c^2}\,\vec{K}$

则得方程

$$\vec{F}=\frac{d\vec{P}}{dt},\quad \vec{P}=m\vec{u},\quad m=\frac{m_0}{\sqrt{1-u^2/c^2}}$$

方程形式与牛顿方程一致。

K_4的第四个分量

$$K_4=\frac{dP_4}{d\tau}=\frac{i}{c}\frac{dW}{d\tau}=\frac{i}{c}\gamma_u\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-u^2/c^2}}\right)$$

$$\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-u^2/c^2}}\right)=\frac{m_0\vec{u}}{\left(1-u^2/c^2\right)^{3/2}}\cdot\frac{d\vec{u}}{dt}$$

$$\vec{F}\cdot\vec{u}=\frac{d\vec{P}}{dt}\cdot\vec{u}=\frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\vec{u}$$

$$=\frac{m_0}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt}\cdot\vec{u}+\frac{m_0u^2}{\left(1-u^2/c^2\right)^{3/2}}\frac{\vec{u}}{c^2}\cdot\frac{d\vec{u}}{dt}$$

$$=\frac{m_0\vec{u}}{(1-u^2/c^2)^{3/2}}\cdot\frac{d\vec{u}}{dt}=\frac{dW}{dt}$$

$$\vec{F}\cdot\vec{u}=\frac{dW}{dt}$$

相对论能量

$$W=\frac{m_0c^2}{\sqrt{1-u^2/c^2}}$$

$$K_4=\frac{i}{c}\gamma_\mu\vec{F}\cdot\vec{u}=\frac{i}{c}\vec{k}\cdot\vec{u}$$

$$\therefore K_\mu=(\vec{K},\frac{i}{c}\vec{K}\cdot\vec{u})$$

相对论力学方程

$$\vec{F} = \frac{d\vec{P}}{dt} \qquad (\vec{P} = m\vec{u})$$ 是相对论协变方程

$$\vec{F} \cdot \vec{u} = \frac{dW}{dt} \qquad (W = mc^2)$$

① 四维速度 $U_\mu = r_u(\vec{u}, ic)$

② 四维动量 $\vec{P} = \left(\vec{P}, \frac{iW}{c}\right)$

③ 四维力 $K_\mu = \left(\vec{K}, \frac{i}{c}\vec{K} \cdot \vec{u}\right) = \gamma_u \left(\vec{F}, \frac{i}{c}\vec{F} \cdot \vec{u}\right)$

洛伦兹力：$\vec{F}=e\left(\vec{E}+\vec{u}\times\vec{B}\right)$

用电磁场张量$F_{\mu\nu}$和 U_ν 表示

$$K_\mu = eF_{\mu\nu}U_\nu$$

$$\vec{K}=\frac{1}{\sqrt{1-u^2/c^2}}e\left(\vec{E}+\vec{u}\times\vec{B}\right)$$

$$\boxed{\vec{F}=\sqrt{1-\frac{u^2}{c^2}}\vec{K}}$$

$$K_1 = eF_{12}U_2 + eF_{13}U_3 + eF_{14}U_4$$

$$= eB_3\gamma\bar{u}_2 - eB_2 r\bar{u}_3 + e\left(\frac{-iE_1}{c}\right)\gamma ic$$

$$= \gamma e\left(\vec{u}\times\vec{B}\right)+e\gamma E_1$$

$$= \frac{1}{\sqrt{1-\beta^2}}e\left(\vec{E}+\vec{u}\times\vec{B}\right)_1$$

$$\boxed{\frac{d\vec{P}}{dt}=e\left(\vec{E}+\vec{u}\times\vec{B}\right)}$$

带电粒子在电磁场中的运动方程。

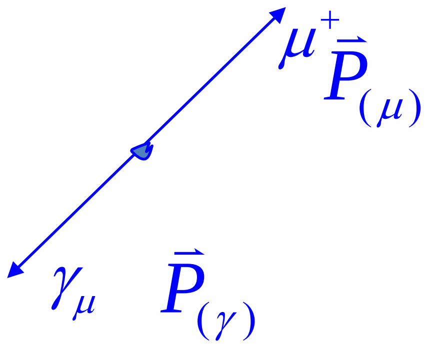
例.带电 π 介质衰变为 μ和中微子。

$$\pi^+ \to \mu^+ + \gamma_\mu$$

各粒子静质量为

$m_\pi = 139.57 MeV/c^2$ $m_\mu = 105.66 MeV/c^2$ $m_\gamma = 0$

求π介子质心系中μ子的动量、能量和速度。



$$\vec{P}_\pi = 0, W_\pi = m_\pi c^2$$

$$W_{(\mu)} = \sqrt{P_{(\mu)}^2 c^2 + m_\mu^2 c^4}$$

$$W_{(\gamma)} = P_{(\gamma)} c$$

由能量和动量守恒定律有

$$\vec{P}_{(\mu)} + \vec{P}_{(\gamma)} = 0 \quad (1), \qquad \sqrt{P_{(\mu)}^2 c^2 + m_\mu^2 c^4} + P_{(\gamma)} c = m_\pi c^2 \quad (2)$$

将(1) 代入 (2)得: $\sqrt{P^2c^2+m_\mu^2c^4}=m_\pi c^2-Pc$

$$\Rightarrow \begin{cases} P_\mu = P_\nu = P = \dfrac{m_\pi^2 - m_\mu^2}{2m_\pi}c \\ W_\mu = m_\pi c^2 - Pc = \dfrac{m_\pi^2 + m_\mu^2}{2m_\pi}c^2 \end{cases} \Rightarrow \begin{array}{l} P = 29.79MeV/c \\ W_\mu = 109.78MeV \end{array}$$

$$\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}=\frac{W_\mu}{m_\mu c^2}=\frac{109.78}{105.66}\equiv 1.0390$$

$$\Rightarrow \quad v_\mu = 0.2714c$$

例3. 讨论带电粒子在均匀恒定常磁场中的运动。

$$\vec{F} = e\vec{u} \times \vec{B} \qquad \frac{d\vec{P}}{dt} = e\vec{u} \times \vec{B}$$

$$\frac{dW}{dt} = (e\vec{u} \times \vec{B}) \cdot \vec{u} = 0 \longrightarrow W = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \text{常数} \longrightarrow |\vec{u}| = \text{常数}$$

表示速度 $\vec{u}$ 方向会改变，但大小不变。

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}\right) = \frac{m_0}{\sqrt{1 - u^2/c^2}} \frac{d\vec{u}}{dt} = e\vec{u} \times \vec{B}$$

即： $\dot{\vec{u}} = \frac{e}{\gamma m_0} \vec{u} \times \vec{B} \quad (1)$

设：$\vec{B} = B\vec{e}_z \quad \vec{u} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z$

$$\Rightarrow \quad \vec{u} \times \vec{B} = B(u_y\vec{e}_x - u_x\vec{e}_y) \quad (2)$$

将 (1) 代入 (2)得:

$$\dot{u}_z = 0 \quad \Rightarrow \quad u_z = 常数$$

∴

令：$\omega = \dfrac{eB}{\gamma m_0}$

$$(1)(2) \Rightarrow \begin{cases} \dot{u}_x = \omega u_y \\ \dot{u}_y = -\omega u_x \end{cases} \quad \Rightarrow \quad \begin{cases} u_x = C\cos\omega t \\ u_y = -C\sin\omega t \end{cases}$$

$$\begin{matrix} u = 常数 \\ u_z = 常数 \end{matrix} \quad \Rightarrow \quad \begin{matrix} u_\perp = C = \sqrt{u^2 - u_z^2} \\ u_x^2 + u_y^2 = u_\perp^2 = C^2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} u_x = u_\perp \cos\omega t \\ u_y = -u_\perp \sin\omega t \end{matrix}$$

若 $u_z = 0$ ，则粒子做匀速圆周运动，角频率

$$\omega = \frac{eB}{\gamma m_0}，其中\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$圆半径 a = \frac{u_\perp}{\omega} = \frac{\gamma m_0 u_\perp}{eB}$$

当 $u_z \neq 0$ 时，粒子做螺旋运动

当 $u << c$ 时 $\gamma \to 1$，则 $\omega = \dfrac{eB}{m_0}$ 与非相对论一致。

在非相对论情形，$\omega = \dfrac{eB}{m_0}$ 与粒子及速度无关。

在相对论情形，$\omega = \dfrac{eB}{\gamma m_0}$ 随粒子速度增大而增大，速度增大 γ 减小，

因此 ω 随粒子速度增大而增大。